Research Proposal

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The overall theme of my research is the study of generalized symmetries, functorial field theories and their interplay, particularly utilising methods from higher category and higher topos theory.

My research goal has two parts: firstly, to assist mathematicians in the functorial field theory community with their quest to formally define quantum field theory; and secondly, to help physicists in the generalized symmetry community understand these symmetries in a rigorous manner and contribute to proving new results in this direction.

About me: I started my academic career in particle phenomenology. Over the last few years, I have turned my attention to mathematical physics and formal high-energy theory, where I have developed a strong working background in ∞-category theory, with a particular emphasis on topos theory. My work in particle phenomenology helps me with my formal/mathematical work to understand what physicists know, and more importantly, what they want to know as well as giving me a cultivated physical intuition.

Personal goal: In the next 3-5 years my goal is to increase my collaboration network within the mathematics and formal physics communities and put my unique combined skill set to use.

Previous and current work

Generalized symmetries and functorial field theories: In the area of generalized symmetries and functorial field theories, I have two papers [GRWTS23] and [GRWTS] (the latter to be published shortly). Both are in collaboration with Ben Gripaios (Cambridge) and Oscar Randal-Williams (Cambridge).

These papers look at invertible generalized symmetries rigorously using the framework of functorial field theory. In [GRWTS23] we gave a definition of *discrete* generalized symmetries of *topological* field theories, and studied the consequences thereof.

The general idea is as follows. Topological field theories of dimension d and with a topological background structure $\mathcal{G}$ collect themselves, by the cobordism hypothesis, into a space X defined as

$$X = \mathrm{Fun}^{\otimes}(\mathrm{Bord}_d^{\mathcal{G}}, \mathcal{C})$$

A point in X, *i.e.* maps $1 \to X$, corresponds to a specific choice in field theory, and two path-connected points are equivalent theories. On X we can act with a topological group G, in a similar way to how a group acts on a set. The higher homotopy groups of G correspond to 'higher-form symmetries' in the physicist's language. The homotopy fixed points of this G-action on X correspond to field theories with a G-symmetry twisted by the action. Nontrivial twistings allow us to recover things like projective representations, and semi-linear representations.

Under this formalism, we provided a classification of symmetries for 1-dimensional and 2-dimensional topological field theories. We also studied gauging, the presence of 't Hooft anomalies, and a variation thereof we called metaphysical 't Hooft anomalies.

The second paper, [GRWTS], studied the global *invertible* generalized symmetries in the *non*-topological setting. Here the topological structure $\mathcal{G}$ is replaced by a geometric one (in line with Grady and Pavlov). The space X is replaced by a smooth space (appropriately defined as a sheaf on the site of manifolds). The topological group G is replaced by a group object in the topos of smooth spaces, and its action on X is replaced by its topos-theoretic counterpart. Importantly, higher homotopy groups of G can be Lie groups, leading to smooth higher-form symmetries. Like before, homotopy fixed points correspond to theories with G-symmetry twisted by the action.

This proposed formalism, allowed us to study symmetries in the case of non-unitary and unitary quantum mechanics (QM), and other simple examples where the smooth space X is known. It allowed us to give a rigorous understanding of many features known by physicists, such as the statement that at most $d-1$-form symmetries can act on d-dimensional field theories.

During this work, we also discovered a new type of anomaly, called a smoothness anomaly. This occurs when a group action associated with a single theory, namely an action on the image of a map $1 \to X$, does not extend to the whole of X. This can be seen to occur even in the simple example of QM, and shows that things can go astray if one has a singular focus on a single theory. In QM, it seems to get rid of examples of group actions which would otherwise seem pathological.

Particle Phenomenology: I have a diverse portfolio of projects in particle phenomenology, with a broad range of different collaborators. As an indication of my work in this field and the sorts of insights it gives me, let me briefly mention my three faculty-independent projects. The first, with Joe Davighi (Zurich), looked at studying the phenomenology of a unified theory based on the $su(4) \oplus sp(6) \oplus sp(6)$ gauge algebra which breaks down to the Standard Model. This extension of the standard model was found in the classification of such extensions I performed in an earlier project. The second, also with Joe Davighi, looked at classifying, using the theory of Lie algebras, abelian extensions of the standard model with semi-simple completions. My third, in collaboration with Maximilian Ruhdorfer (Cornell) and Andrew Gomes (Cornell), involved a Mathematica program designed to classify all semi-simple extensions based on an arbitrary input theory.

Planned projects

My planned future work revolves around extending the results in [GRWTS23] and [GRWTS]. Below, I present a summary of the projects I would like to complete within the next 3-5 years. All the projects are aligned with my overarching research goal outlined at the beginning of this proposal.

Understanding smoothness anomalies: As mentioned above, in [GRWTS] we discovered a new type of anomaly, termed smoothness anomalies. This project would aim to better understand smoothness anomalies by studying and classifying them, in a series of simple examples, such as QM. Specific questions of interest include: Does the presence of such an anomaly prohibit the gauging of the associated symmetry? Does it impose constraints on how physicists conventionally utilize this symmetry?

To address these queries, one methodological approach I propose is to transition from the topos of sheaves on manifolds to that of sheaves on topological spaces, or analogous sites. This change in arena reframes the issue of smoothness as one of continuity, potentially offering new insights into the characteristics and implications of smoothness anomalies.

Symmetries through geometric structures: In the study of generalized symmetries, along the lines of [GRWTS23] and [GRWTS], one critical bottleneck lies in determining a nice description of the (smooth) space X of field theories. In particular, the literature is sparse of such examples. This may seem to obstruct the study of symmetries since one needs to find G-actions on X as discussed above. However, physicists often act not on X (since in most cases they don't know this) but rather on the background structures, such as fields and metrics (pre-quantization). This project aims to formalise this idea. Indeed, we will look at what it means to act on the background structure $\mathcal{G}$, and how this action can be translated to X.

I envision two complementary pathways to achieve the goals of this project. The first leverages the insights of Grady and Pavlov concerning the geometric cobordism hypothesis, while the second employs Factorization Homology as a distinct approach to the topological cobordism hypothesis. Ideally, both methods should be explored for a more comprehensive understanding.

A non-trivial aspect of this project involves identifying the appropriate topos theretic context for housing the group G. The group's behaviour can differ significantly depending on whether it resides in the category of geometric structures or the category of smooth spaces.

Integrating out fields: Given a background field/structure, $\mathcal{G}$, a physicist may wish to perform one of two operations. The first is quantization, where one sums over all permissible configurations of $\mathcal{G}$. Although a formal methodology for this remains elusive, it is a subject of ongoing research by multiple groups. The second operation is to "integrate out" the background field, meaning to solve its classical equations of motion and then substitute these solutions back into the field theory.

The aim of this proposed project is to develop a rigorous framework for performing the second operation in the context of functorial field theory. Contrary to quantization, this process is conceptually simpler as it sidesteps enigmatic elements like the path integral. The initial approach will be to concentrate on one-dimensional field theories that have a background structure consisting of a map to a target manifold and a Riemannian metric. By integrating out the mapping structure, we aim to recover the principles of classical mechanics. Subsequent to this, I plan to examine how symmetries may be influenced by this "integrating out" operation.

External non-invertible generalized symmetries: The definition of the (smooth) space X given above does not have information about non-invertible defects, only the invertible ones (given by loops in the space). There are research collaborations attempting to determine how to generalize X to include non-invertible defects in both the topological and smooth cases. Independent of these specific advancements - some of which this project may contribute to - arises the question: if X is augmented to include non-invertibility, how should we conceptualize non-invertible generalized symmetries in a similar manner to [GRWTS23] and [GRWTS]? More pointedly, what mathematical object should replace the group G? How does this newly introduced structure act on X and yield homotopy fixed points?

The most plausible resolution to these questions seems to lie in the domain of higher algebras. Specifically, the group G would be supplanted by a higher algebra of some variety.

Much of the existing literature in physics on generalized symmetries is centred around the theme of non-invertibility. As such, the proposed research serves as a crucial step in bridging the divide between the mathematical and physical communities, facilitating a more unified understanding of generalized symmetries.

Noether's theorem: The question of the utility of symmetries in field theories is as much philosophical as it is mathematical. More precisely, one might ask: what *can* symmetries be employed for? A classic application lies in the identification of conserved quantities through Noether's theorem, which is applicable to smooth symmetries of theories. Building upon [GRWTS], we should be well-positioned to understand Noether's theorem in a rigorous mathematical language.

A possible methodology is to use an area within higher topos theory called differential cohesion. There may be a way to also connect this work with factorization algebras, a topic currently being discussed by parts of the community.

References:

[GRWTS] Ben Gripaios, Oscar Randal-Williams, and Joseph Tooby-Smith. Smooth generalized symmetries of quantum field theories. ?? to appear.

[GRWTS23] Ben Gripaios, Oscar Randal-Williams, and Joseph Tooby-Smith. Generalized symmetries of topological field theories. *JHEP*, 03:087, 2023.